FERMIONIC FORM AND BETTI NUMBERS

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Abstract. We state a conjectural relationship between the fermionic form [HKOTY] and the Betti numbers of a Grassmannian over a preprojective algebra or, equivalently, of a lagrangian quiver variety.

1. Notation. We fix a graph of type ADE with set of vertices I. Let E be a $\mathbf{R}$-vector space with a basis $(\alpha_i)_{i\in I}$ and a positive definite symmetric bilinear form $(,):E\times E\longrightarrow\mathbf{R}$ given by $(\alpha_i,\alpha_i)=2$, $(\alpha_i,\alpha_j)=-1$ if i,j are joined in the graph, $(\alpha_i,\alpha_j)=0$ if $i\neq j$ are not joined in the graph. Let $(\varpi_i)_{i\in I}$ be the basis of E defined by $(\varpi_i,\alpha_j)=\delta_{i,j}$. For $\xi\in E$ define ${}^i\xi,{}_i\xi$ in $\mathbf{R}$ by

$$\xi=\sum_i({}^i\xi)\varpi_i=\sum_i({}_i\xi)\alpha_i.$$

Let $P=\{\xi\in E|{}^i\xi\in\mathbf{Z}\quad\forall i\in I\}$, $P^+=\{\xi\in E|{}^i\xi\in\mathbf{N}\quad\forall i\in I\}$. Let $\rho=\sum_i\varpi_i\in P^+$. We consider the usual partial order on P:

$$\xi\le\xi'\Leftrightarrow{}_i\xi'-{}_i\xi\in\mathbf{N}\text{ for all }i.$$

For $i\in I$ define $s_i:E\longrightarrow E$ by $s_i(\xi)=\xi-(\xi,\alpha_i)\alpha_i$. Let W be the (finite) subgroup of $GL(E)$ generated by $\{s_i|i\in I\}$. Let $\mathbf{Z}[P]$ be the group ring of P with obvious basis $([\xi])_{\xi\in P}$. For $\xi\in P^+$ define $V_\xi\in\mathbf{Z}[P]$ by Weyl's formula

$$\sum_{w\in W}\det(w)[w(\xi+\rho)]=V_\xi\sum_{w\in W}\det(w)[w(\rho)].$$

2. The fermionic form [HKOTY]. Let q be an indeterminate. For $p,m\in\mathbf{N}$ define

$$\begin{bmatrix}p+m\\m\end{bmatrix}=\frac{(q^{p+1}-1)(q^{p+2}-1)\dots(q^{p+m}-1)}{(q-1)(q^2-1)\dots(q^m-1)}\in\mathbf{Z}[q].$$

Let $\nu=\{\nu_k^{(i)}\in\mathbf{N}|i\in I,k\ge1)$ where all but finitely many $\nu_k^{(i)}$ are zero. Let $\lambda\in P^+$. In [HKOTY, 4.3] a "fermionic form" $M(\nu,\lambda,q)$ (or $M(W,\lambda,q)$ in the

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notation of *loc.cit.*) is attached to ν, λ. This is a q-analogue of an expression which first appeared in Kirillov and Reshetikhin [KR]. For $q = 1$ it conjecturally gives the multiplicities in certain representations of an affine quantum group when restricted to the ordinary quantum group. In [Kl], Kleber rewrites the formula of [KR] in the form of a computationally efficient algorithm. (In his paper, it is assumed that one of the $\nu_k^{(i)}$ is 1 and the other s are 0 but, as he pointed out to me, the same procedure works in general for $q = 1$.)

In the remainder of this note we assume that

$$\nu_k^{(i)} = 0 \text{ for } i \in I, k \geq 2.$$

In this case we identify ν with the element of P^+ such that ${}^i\nu = \nu_1^{(i)}$ for all i. By definition,

(a)
$$M(\nu, \lambda, q) = \sum_{\{m\}} q^{c(\{m\})} \prod_{i \in I; k \geq 1} \begin{bmatrix} p_k^{(i)} + m_k^{(i)} \\ m_k^{(i)} \end{bmatrix},$$

$$c(\{m\}) = \frac{1}{2} \sum_{i,j \in I} (\alpha_i, \alpha_j) \sum_{k,l \geq 1} \min(k,l) m_k^{(i)} m_l^{(j)} - \sum_{i \in I} \sum_{k \geq 1} {}^i\nu m_k^{(i)},$$

$$p_k^{(i)} = {}^i\nu - \sum_{j \in I} (\alpha_i, \alpha_j) \sum_{l \geq 1} \min(k,l) m_l^{(j)},$$

where the sum $\sum_{\{m\}}$ is taken over $\{m_k^{(i)} \in \mathbf{N} | i \in I, k \geq 1\}$ satisfying $p_k^{(i)} \geq 0$ for $i \in I, k \geq 1$ and

$$\sum_{i \in I} \sum_{k \geq 1} k m_k^{(i)} \alpha_i = \nu - \lambda$$

for $i \in I$. We rewrite this by extending the method of [Kl] to the q-analogue; we obtain

(b)
$$M(\nu, \lambda, q) = \sum_{\omega} q^{c(\omega)} \prod_{i \in I; k \geq 1} \begin{bmatrix} {}^i\omega_k + {}_i\mu_k \\ {}_i\mu_k \end{bmatrix}$$

sum over all sequences ω in P^+ of the form

$$\nu = \omega_0 > \omega_1 > \omega_2 > \cdots > \omega_s = \omega_{s+1} = \omega_{s+2} = \cdots = \lambda$$

such that

$$\omega_0 - \omega_1 \geq \omega_1 - \omega_2 \geq \omega_2 - \omega_3 \geq \dots$$

that is,

$$\mu_k = \omega_{k-1} - 2\omega_k + \omega_{k+1} \geq 0 \text{ for } k \geq 1, \quad \mu_k = 0 \text{ for } k \gg 0,$$

and

$$c(\omega) = \frac{1}{2}\sum_{k\geq 1}(X_k, X_k) - (\nu, X_1)$$

where

$$X_k = \omega_{k-1} - \omega_k \text{ for } k \geq 1.$$

The connection between (a) and (b) is as follows: in terms of the data in (a) we have

$$\omega_k = \sum_i p_k^{(i)}\varpi_i, \quad \mu_k = \sum_i m_k^{(i)}\alpha_i.$$

Since $\mu_k = X_k - X_{k+1}$ for $k \geq 1$, we have for $i, j \in I$:

$$\begin{aligned}
&\sum_{k,l\geq 1} \min(k,l) m_k^{(i)} m_l^{(j)} = \sum_{k,l\geq 1} \min(k,l)({}_iX_k - {}_iX_{k+1})({}_jX_l - {}_jX_{l+1}) \\
&= \sum_{k,l\geq 1} \min(k,l)({}_iX_k({}_jX_l) - {}_iX_{k+1}({}_jX_l) - {}_iX_k({}_jX_{l+1}) + {}_iX_{k+1}({}_jX_{l+1})) \\
&= \sum_{k,l\geq 1} (\min(k,l) - \min(k-1,l) - \min(k,l-1) + \min(k-1,l-1)){}_iX_k({}_jX_l) \\
&= \sum_{k\geq 1} {}_iX_k({}_jX_k),
\end{aligned}$$

$$\sum_{k\geq 1} {}^i\nu m_k^{(i)} = \sum_{k\geq 1} {}^i\nu({}_iX_k - {}_iX_{k+1}) = ({}^i\nu)({}_iX_1),$$

hence $c(\{m\}) = c(\omega)$.

The following result is stated without proof in [HKOTY].

Lemma 3. $M(\nu, \lambda, q) \in \mathbf{N}[q^{-1}]$.

Let ω be as in Sec.2. The product of q-binomial coefficients in the term corresponding to ω is a polynomial in q of degree

$$\begin{aligned}
N &= \sum_{i\in I; k\geq 1} {}^i\omega_k({}_i\mu_k) = \sum_{k\geq 1}(\omega_k, \mu_k) \\
&= \sum_{k\geq 1}(\nu - X_1 - X_2 - \cdots - X_k, X_k - X_{k+1}) = (\nu, X_1) - \sum_{k\geq 1}(X_k, X_k).
\end{aligned}$$

It is enough to show that $c(\omega) + N \leq 0$. We have

$$c(\omega) + N = -\frac{1}{2}\sum_{k\geq 1}(X_k, X_k)$$

and this is clearly ≤ 0.

Lemma 4. *Let $\xi \in P^+$ and let $\eta \in P$ be such that $\eta \geq 0$. Then $(\xi, \eta) \geq 0$.*

This is obvious.

Lemma 5. *If $\nu \geq \lambda$ then $M(\nu, \lambda, q) = q^{-(\nu,\nu)/2+(\lambda,\lambda)/2} +$ strictly larger powers of q.*

Let ω be as in Sec.2. We show that

(a) $$c(\omega) \geq -(\nu, \nu)/2 + (\lambda, \lambda)/2$$

that is,

$$\frac{1}{2}\sum_{k\geq 1}(X_k, X_k) - (\nu, X_1) - \frac{1}{2}(\nu - \lambda, \nu - \lambda) + (\nu, \nu - \lambda) \geq 0.$$

Since $\nu - \lambda = X_1 + X_2 + X_3 + \dots$, this is equivalent to

(b) $$(\nu, X_2 + X_3 + \dots) - \sum_{1\leq k<l}(X_k, X_l) \geq 0.$$

Applying Lemma 4 to $\xi = \nu - X_1 - X_2 - \dots - X_{k+1} = \omega_{k+1}$, $\eta = X_{k+1}$, we obtain

$$(\nu - X_1 - X_2 - \dots - X_{k+1}, X_{k+1}) \geq 0.$$

Adding these inequalities over all $k \geq 1$ we obtain

$$\sum_{k\geq 1}(\nu - X_1 - X_2 - \dots - X_k - X_{k+1}, X_{k+1}) \geq 0,$$

that is,

$$(\nu, X_2 + X_3 + \dots) - \sum_{1\leq k<l\geq 1}(X_k, X_l) \geq \sum_{k\geq 2}(X_k, X_k) \geq 0.$$

Thus, (b) hence (a) are proved. This proof shows also that the inequality (a) is strict unless ω satisfies $X_2 = X_3 = \dots = 0$. If this last condition is satisfied then ω is the sequence $\nu = \omega^0 \geq \omega^1 = \omega^2 = \dots = \lambda$ and (a) is an equality. The lemma is proved.

6. Inversion. Define $M^*(\nu, \lambda, q) \in \mathbf{Z}[q^{-1}]$ for any $\nu, \lambda \in P^+$ by the requirement that the matrix $(M^*(\nu, \lambda, q))_{\mu,\lambda}$ is inverse to the matrix $(M(\nu, \lambda, q))_{\mu,\lambda}$ (which is lower triangular with 1 on diagonal). Thus, $M^*(\nu, \nu, q) = 1$ and $\sum_{\lambda\in P^+} M^*(\nu, \lambda, q)M(\lambda, \xi, q) = 0$ for any $\nu > \xi$ in P^+. There is some evidence that the matrix M^* is simpler than M. For example, in type A_1, we have

$$M^*(\nu, \lambda, 1) = (-1)^{{}_i(\nu-\lambda)}\binom{{}^i\lambda + {}_i(\nu - \lambda)}{{}_i(\nu - \lambda)}$$

for any $\nu \geq \lambda$ in P^+.

7. Path algebra. Let $\mathbf{I}$ be the set of all sequences $i_1, i_2, \dots, i_s$ (with $s \geq 1$) in I such that i_k, i_{k+1} are joined for any $k \in [1, s-1]$. Let $\mathcal{F}$ be the $\mathbf{C}$-vector space spanned by elements $[i_1, i_2, \dots, i_s]$ corresponding to the various elements of $\mathbf{I}$. We regard $\mathcal{F}$ as an algebra in which the product $[i_1, i_2, \dots, i_s][j_1, j_2, \dots, j_{s'}]$ is equal to $[i_1, i_2, \dots, i_s, j_2, \dots, j_{s'}]$ if $i_s = j_1$ and is zero, otherwise. For $i \in I$, let $\vartheta_i = \sum_j [iji]$ where j runs over the elements of I that are joined with i. For $i, j \in I$ let $\mathcal{F}_{ij}$ be the subspace of $\mathcal{F}$ spanned by the elements $[i_1, i_2, \dots, i_s]$ with $i_1 = i, i_s = j$. For $u \in \mathbf{Z}$ let $\mathcal{F}^u$ be the subspace of $\mathcal{F}$ spanned by the elements $[i_1, i_2, \dots, i_s]$ with $s = u + 1$. (For $u < 0$ we have $\mathcal{F}^u = 0$.) Let $\mathcal{F}^u_{ij} = \mathcal{F}_{ij} \cap \mathcal{F}^u$. We have

$$\mathcal{F} = \oplus_{i,j} \mathcal{F}_{ij}, \mathcal{F} = \oplus_u \mathcal{F}^u, \mathcal{F} = \oplus_{i,j;u} \mathcal{F}^u_{ij}.$$

Let $\mathcal{I}$ be the two-sided ideal of $\mathcal{F}$ generated by the elements ϑ_i ($i \in I$). The quotient algebra $\mathbf{P} = \mathcal{F}/\mathcal{I}$ has finite dimension over $\mathbf{C}$ [GP]. Let $\mathbf{P}_{ij}, \mathbf{P}^u, \mathbf{P}^u_{ij}$ be the image of $\mathcal{F}_{ij}, \mathcal{F}^u, \mathcal{F}^u_{ij}$ in $\mathbf{P}$. We have

$$\mathbf{P} = \oplus_{i,j} \mathbf{P}_{ij}, \mathbf{P} = \oplus_u \mathbf{P}^u, \mathbf{P} = \oplus_{i,j;u} \mathbf{P}^u_{ij}.$$

Let $\mathbf{D}$ a finite dimensional $\mathbf{C}$-vector with a given direct sum decomposition $\mathbf{D} = \oplus_{i \in I} \mathbf{D}_i$. Then $\mathbf{D}^\dagger = \oplus_{i,j} \mathbf{P}_{ij} \otimes \mathbf{D}_j$ is a left $\mathbf{P}$-module in an obvious way (in fact a projective $\mathbf{P}$-module of finite dimension over $\mathbf{C}$). Let $\nu = \sum_{i \in I} \dim \mathbf{D}_i \varpi_i \in P^+$. Let $\mathrm{Grass}_{\mathbf{P}}(\mathbf{D}^\dagger)$ be the algebraic variety consisting of all $\mathbf{P}$-submodules of $\mathbf{D}^\dagger$. We have a partition

$$\mathrm{Grass}_{\mathbf{P}}(\mathbf{D}^\dagger) = \sqcup_{\xi \in P} \mathrm{Grass}_{\mathbf{P},\xi}(\mathbf{D}^\dagger)$$

where $\mathrm{Grass}_{\mathbf{P},\xi}(\mathbf{D}^\dagger)$ consists of all $\mathbf{P}$-submodules $\mathcal{V}$ such that $\sum_i \dim([i]\mathbf{D}^\dagger)/[i]\mathcal{V})\alpha_i = \nu - \xi$. Then

Conjecture A. *Let $q^{1/2}$ be an indeterminate. For any $\xi \in P$ we have*

$$\text{(a)} \qquad \sum_{s \in \mathbf{N}} \dim H^s(\mathrm{Grass}_{\mathbf{P},\xi}(\mathbf{D}^\dagger)) q^{s/2} = \sum_{\lambda \in P^+} (\xi : V_\lambda) q^{(\nu,\nu)/2 - (\xi,\xi)/2} M(\nu, \lambda, q)$$

where $(\xi : V_\lambda)$ is the coefficient in ξ in V_λ and $H^s()$ denotes ordinary cohomology with coefficients in a field.

Since $(\xi : V_\lambda)$ and (ξ, ξ) are W-invariant in ξ, we see that the right hand side of (a) is W-invariant in ξ. The analogous property of the left hand side of (a) is known to be true. (See [L2].)

In [L1] it is shown that $\mathrm{Grass}_{\mathbf{P},\xi}(\mathbf{D}^\dagger)$ is isomorphic to a (lagrangian) quiver variety defined in Nakajima [NA] and, conversely, all such quiver varieties are obtained. Thus the conjecture above gives at the same time the Betti numbers of quiver varieties.

Assuming the conjecture, we show that $\mathrm{Grass}_{\mathbf{P},\xi}(\mathbf{D}^\dagger)$ is connected if $(\xi : V_\nu) \neq 0$ (an expected but not yet proved property of quiver varieties). We may assume

that $\xi \in P^+, \xi \le \nu$. It suffices to show that $\dim H^0(\mathrm{Grass}_{\mathbf{P},\xi}(\mathbf{D}^\dagger)) = 1$ or that the constant term of

$$\sum_{\lambda \in P^+} (\xi : V_\lambda) q^{(\nu,\nu)/2-(\xi,\xi)/2} M(\nu,\lambda,q)$$

is 1. By Lemma 5, the constant term of the term corresponding to $\lambda = \xi$ is 1. Consider now the term corresponding to $\lambda \ne \xi$; we show that its constant term is 0. We may assume that $\lambda \le \nu$ and $(\xi : V_\lambda) \ne 0$ so that $\xi < \lambda$. By Lemma 5, $q^{(\nu,\nu)/2-(\xi,\xi)/2} M(\nu,\lambda,q)$ is of the form

$$q^{(\nu,\nu)/2-(\xi,\xi)/2)} q^{-(\nu,\nu)/2+(\lambda,\lambda)/2} + \text{ strictly larger powers of } q.$$

Since $(\xi,\xi) < (\lambda,\lambda)$ for any λ, ξ in P^+ such that $\xi < \lambda$, our assertion is established.

The same argument shows that the right hand side of (a) is in $\mathbf{N}[q]$.

Assuming again that $\xi \in P^+, \xi \le \nu$ we show that the polynomial in q given by the right hand side of (a) has degree $s = (\nu,\nu)/2 - (\xi,\xi)/2$. The term corresponding to $\lambda = \nu$ is $(\xi : V_\nu) q^s$ where $(\xi : V_\nu) > 0$. Consider now the term corresponding to $\lambda \ne \nu$. We may assume that $\lambda < \nu$. It suffices to show that in this case $M(\nu,\lambda,q) \in q^{-1}\mathbf{Z}[q^{-1}]$. This follows from the proof of Lemma 3. Our assertion is established. Note that this is compatible with the conjecture since the dimension of $\mathrm{Grass}_{\mathbf{P},\xi}(\mathbf{D}^\dagger)$ is known to be equal to $(\nu,\nu)/2 - (\xi,\xi)/2$.

In the A_1 case, the left hand side of (a) is a q-binomial coefficient; the right hand side can be computed by results in [Ki],[KSS]; the conjecture holds in this case.

8. The conjecture implies that

$$\text{(a)} \qquad \chi(\mathrm{Grass}_{\mathbf{P}}(\mathbf{D}^\dagger)) = \sum_{\lambda \in P^+} \dim(V_\lambda) M(\nu,\lambda,1)$$

where χ denotes Euler characteristic and $\dim(V_\lambda) = \sum_{\xi \in P} (\xi : V_\lambda)$. Let $f(\nu)$ (resp. $g(\nu)$) be the left (resp. right) hand side of (a). According to [HKOTY], it is expected that $g(\nu + \nu') = g(\nu)g(\nu')$ for any $\nu, \nu' \in P^+$. The corresponding identity $f(\nu + \nu') = f(\nu)f(\nu')$ is known [L3, 3.20].

9. We can partition I into two disjoint subsets I^0, I^1 so that no two vertices in I^0 are joined and no two vertices in I^1 are joined. For any $u \in \mathbf{Z}$ let

$${}^u\mathbf{D}^\dagger = \oplus_{i \in I^\delta, j \in I} (\mathbf{P}^u_{ij} \oplus \mathbf{P}^{u-1}_{ij}) \otimes \mathbf{D}_j$$

where $\delta \in \{0,1\}$ is defined by $u = \delta \mod 2$. We have $\mathbf{D}^\dagger = \oplus_u {}^u\mathbf{D}^\dagger$. Consider the $\mathbf{C}^*$-action $t, d \mapsto td$ on $\mathbf{D}^\dagger$ with weight u on ${}^u\mathbf{D}^\dagger$. This action is compatible with the $\mathbf{P}$-module structure in the following sense: $t(pd) = (tp)(td)$ for $t \in \mathbf{C}^*, p \in \mathbf{P}, d \in \mathbf{D}^\dagger$ where tp is given by the $\mathbf{C}^*$-action on $\mathbf{P}$ (through algebra automorphisms) for which $\mathbf{P}^u$ has weight u. Hence we have an induced $\mathbf{C}^*$-action on $\mathrm{Grass}_{\mathbf{P},\xi}(\mathbf{D}^\dagger)$ for any ξ. Let $\mathrm{Grass}'_{\mathbf{P},\xi}(\mathbf{D}^\dagger)$ be the fixed point set of this $\mathbf{C}^*$-action (a smooth variety). It consists of all $\mathcal{V} \in \mathrm{Grass}_{\mathbf{P},\xi}(\mathbf{D}^\dagger)$ such that $\mathcal{V} = \oplus_u (\mathcal{V} \cap {}^u\mathbf{D}^\dagger)$.

Conjecture B. *For any $\xi \in P$ we have*

$$\sum_{s\in\mathbf{N}} \dim H^s(\mathrm{Grass}'_{\mathbf{P},\xi}(\mathbf{D}^\dagger))q^{s/2} = \sum_{\lambda\in P^+} (\xi : V_\lambda) M(\nu,\lambda,q^{-1}).$$

One can show that this is equivalent to Conjecture A.

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